JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **146**, 397–407 (1990)

On a Quasi-regular Lagrange Problem

HANS SAGAN

Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27695-8205

Submitted by Frank H. Clarke

Received February 8, 1988

Following a historical trail that begins with J. Radon's 1910 dissertation, one is led, almost perchance, to a non-trivial Lagrange problem which displays a number of remarkable features: it is quasi-regular, an explicit representation of its extremals can be obtained by simple integrations, it admits—in spite of having transcendental constraints—linear variations, and, finally, its extremals do not even yield a weak extremum. © 1990 Academic Press, Inc.

1. INTRODUCTION

J. Radon (who was born 100 years ago on December 16, 1887) considered in his 1910 Vienna dissertation [1] homogeneous variational problems with integrands that contain, next to x, y, $\dot{x}$, $\dot{y}$, also $\ddot{x}$ and $\ddot{y}$, by introducing the parameter-invariants θ and κ, the angle of the tangent line and the curvature, and the arclength s as parameter to arrive at the problem of minimizing

$$\int_{s_0}^{s_1} f(x, y, \theta, \kappa)\, ds,$$

where the coordinates and directions of the endpoints of the solution $x = x(s)$, $y = y(s)$ are prescribed. In a generalization of Weierstrass' problem where the integrand only depends on x, y, $\dot{x}$, $\dot{y}$, he establishes necessary conditions for a minimum and sufficient conditions for weak and, what Zermelo called, semi-strong minima. He applied his theory to problems where the integrand only depends on κ and showed that in such a case one may obtain the extremals by simple quadratures.

The problem was later taken up by W. Blaschke who showed [2] that the extremals for the problem in three dimensions with the integrand $\sqrt{\kappa}$ are generalized helices (what Emil Mueller called "Boeschungslinien") and

0022-247X/90 $3.00

that among problems where the integrand depends on κ only, the ones with the integrand $\sqrt{\kappa}$ are essentially the only ones with generalized helices as extremals.

R. Irrgang took it from there and, encouraged by J. Radon, considered in his 1933 Breslau dissertation [3] problems where the integrand depends on the curvature κ and the torsion τ. He treated such problems as Lagrange problems with Fresnet's formulas and the definition of the tangent vector as constraints. Such problems are singular. He succeeded in finding the extremals for the case where the integrand is $\sqrt{\tau}$. They turn out to be helical curves that move away from their axis exponentially. He concludes his paper with the enigmatic remark that the excess function cannot be established for this case and that the nature of the extremum cannot be ascertained.

Quite possibly prompted by Irrgang's difficulties and with the expectation of clearing them up, J. Radon studied in 1937 [4] Lagrange problems where the derivative of one of the unknown functions is missing—which renders the problems singular—and sketched some necessary and sufficient conditions. Specifically, he posed the problem of finding a curve in $(x, y_1, y_2, ..., y_n)$-space that joins two lines that are parallel to the y_n-axis and is such that

$$\int_a^b f(x, y_1, y_2, ..., y_n, y_1', y_2', ..., y_{n-1}')\, dx$$

is rendered a minimum under the constraints

$$\varphi_\rho(x, y_1, y_2, ..., y_n, y_1', y_2', ..., y_{n-1}') = 0, \qquad \rho = 1, 2, ..., m < n-1,$$

where it is assumed that $\operatorname{rank}(\partial\varphi_\rho/\partial y_i') = m$. Since

$$\Delta_1 = \frac{\partial(F_{y_1'}, F_{y_2'}, ..., F_{y_n'}, \varphi_1, \varphi_2, ..., \varphi_m)}{\partial(y_1', y_2', ..., y_n', \lambda_1, \lambda_2, ..., \lambda_m)} = 0 \tag{1}$$

with

$$F = f + \sum_{\rho=1}^{m} \lambda_\rho \varphi_\rho$$

the problem is singular. To carry out the modified Legendre transformation

$$F_{y_i'} = p_i, \qquad F_{y_n} = 0, \qquad \varphi_\rho = 0, \qquad H = \sum_{i=1}^{n-1} y_i' F_{y_i'} - F$$

and avail himself of the advantages that the Euler–Lagrange equations in canonical form present, he assumes that

$$\Delta_2 = \frac{\partial(F_{y_1'}, F_{y_2'}, \ldots, F_{y_{n-1}'}, F_{y_n}, \varphi_1, \varphi_2, \ldots, \varphi_m)}{\partial(y_1', y_2', \ldots, y_{n-1}', y_n, \lambda_1, \lambda_2, \ldots, \lambda_m)} \neq 0. \tag{2}$$

Singular Lagrange problems that satisfy (2) are called *quasi-regular* or *singular of the Radon Type.*

Radon suggests [4, p. 224] that examples of such problems may be obtained if one considers variational problems with integrands where the third derivatives of the unknown functions y, z only appear in the combination $y''z''' - y'''z''$ and where new variables are introduced by means of

$$y_1 = y', \qquad z_1 = z' \tag{3}$$

$$\begin{aligned} y_1' &= z_2 \cos y_2 \\ z_1' &= z_2 \sin y_2 \end{aligned} \tag{4}$$

and he refers to Irrgang's problem as a special case therof. While this is true for some cases, it is not true for the problem with $\sqrt{\tau}$ as integrand for which Irrgang could not determine the nature of the extremum. Indeed, were one to formulate the latter problem as a non-homogeneous problem, one would be led to the integrand

$$\sqrt{\frac{y''z''' - y'''z''}{(y'z'' - z'y'')^2 + y''^2 + z''^2}}\sqrt{1 + y'^2 + z'^2}.$$

Introducing new variables according to (3) and (4) leads to the Lagrange problem of minimizing

$$\int_a^b \sqrt{\frac{y_2'}{1 + (y_1 \sin y_2 - z_1 \cos y_2)^2}}\sqrt{1 + y_1^2 + z_1^2}\, dx$$

under the constraints (3) and (4). Since z_2' is missing, one row in Δ_1 as defined in (1) is missing and the problem is singular. A simple calculation of Δ_2 as defined in (2) reveals, however, that also $\Delta_2 = 0$ and that the problem is not quasi-regular. One can convince oneself quite easily that this is generally the case when the integrand is a function of τ only. A somewhat modified problem, which is quasi-regular, will be discussed in the next section.

2. Formulation of the Problem

We wish to find a curve C which minimizes

$$\int_0^a \sqrt{y''z''' - y'''z''}\, dx,$$

where we assume w.l.o.g. that the lower integration limit is 0, where we will assume that $a > 0$ is small enough to prevent denominators from becoming 0, square roots from becoming imaginary, or the transformation (4) from losing its bijectivity, and where we leave the formulation of the boundary conditions for later.

If we introduce the new variables y_1, z_1, y_2, z_2 according to (3) and (4), we obtain the problem of minimizing

$$I_C = \int_0^a z_2 \sqrt{y_2'}\, dx \tag{5}$$

under the constraints (3), (4). Since z_2' is missing, the problem is singular but the computation of Δ_2 as defined in (2) yields

$$\Delta_2 = \frac{1}{4y_2'}$$

making the problem quasi-regular and J. Radon's theory applicable. Accordingly, we may seek an extremal that joins two parallel lines

$$x = 0, \qquad y = y^{(1)}, \qquad y_1 = y_1^{(1)}, \qquad y_2 = y_2^{(1)}, \qquad z = z^{(1)}, \qquad z_1 = z_1^{(1)}$$

and

$$x = a, \qquad y = y^{(2)}, \qquad y_1 = y_1^{(2)}, \qquad y_2 = y_2^{(2)}, \qquad z = z^{(2)}, \qquad z_1 = z_1^{(2)}$$

in $(x, y, y_1, y_2, z, z_1, z_2)$-space. This means that, in the original formulation, one may prescribe the boundary values of x, y, y', z', and the ratio y''/z''. We obtain an enormous simplification if we consider instead the problem where the beginning point or the endpoint in (x, y, z)-space varies in the plane $x = 0$, or $x = a$, respectively. Intuitively, this suggests that the constraints (2) may be dropped. This is indeed borne out by the transversality conditions for the Lagrange problem (5, p. 266) which, together with the Euler–Lagrange equations effect the elimination of y, z from the problem altogether. Accordingly, we will consider the problem of minimizing (5) under the constraints (4).

3. The Euler–Lagrange Equations and the Excess Function

With

$$F = z_2\sqrt{y_2'} + \lambda(y_1' - z_2 \cos y_2) + \mu(z_1' - z_2 \sin y_2),$$

we obtain for the Euler–Lagrange equations

$$\begin{aligned} F_{y_i} - (d/dx)\, F_{y_i'} &= 0, \qquad i = 1, 2 \\ F_{z_1} - (d/dx)\, F_{z_1'} &= 0 \\ F_{z_2} &= 0 \end{aligned}$$

for the problem (4), (5)

$$\lambda' = 0, \qquad \mu' = 0 \tag{6}$$

$$(d/dx)(z_2/2\sqrt{y_2'}) = \lambda z_2 \sin y_2 - \mu z_2 \cos y_2 \tag{7}$$

$$y_2' = (\lambda \cos y_2 + \mu \sin y_2)^2 \tag{8}$$

which are to be solved in conjunction with the constraints (4). From (6), λ, μ are constants, not both of which can vanish. Otherwise, $F_{y_2'}$ would not be defined. We obtain from (8)

$$\frac{dy_2}{(\lambda^2 + \mu^2)\cos^2(y_2 + \delta)} = dx \qquad \text{for} \quad \delta = \tan^{-1}\frac{\mu}{\lambda}$$

and hence,

$$y_2 = \tan^{-1}[(\lambda^2 + \mu^2)(x + \alpha)] + \tan^{-1}\frac{\mu}{\lambda} \tag{9}$$

for arbitrary α. In order to solve (7), we use

$$\begin{aligned} \cos y_2 &= (\lambda - \mu(\lambda^2 + \mu^2)(x + \alpha))/\sqrt{\lambda^2 + \mu^2}\,\sqrt{1 + [(\lambda^2 + \mu^2)(x + \alpha)]^2} \\ \sin y_2 &= (\mu + \lambda(\lambda^2 + \mu^2)(x + \alpha))/\sqrt{\lambda^2 + \mu^2}\,\sqrt{1 + [(\lambda^2 + \mu^2)(x + \alpha)]^2} \end{aligned} \tag{10}$$

and after substituting in (7) and some cumbersome manipulations we obtain that

$$\frac{dz_2}{z_2} = \frac{(\lambda^2 + \mu^2)^2\,(x + \alpha)}{1 + [(\lambda^2 + \mu^2)(x + \alpha)]^2}\,dx$$

with the solution

$$z_2 = \beta\sqrt{1 + [(\lambda^2 + \mu^2)(x + \alpha)]^2} \tag{11}$$

for arbitrary β. Finally, from the constraining equations (4), in view of (9), we obtain that

$$\begin{aligned} y_1' &= \frac{\beta}{\sqrt{\lambda^2+\mu^2}} (\lambda - \mu(\lambda^2+\mu^2)(x+\alpha)), \\ z_1' &= \frac{\beta}{\sqrt{\lambda^2+\mu^2}} (\mu + \lambda(\lambda^2+\mu^2)(x+\alpha)) \end{aligned} \tag{12}$$

from which y_1, z_1, and in turn, y, z for a given beginning point or a given endpoint, can be easily obtained.

According to Radon [4, p. 223], the excess function for a problem of this type is to be defined as

$$\begin{aligned} &E(x, y_1, y_2, z_1, z_2, \bar{z}_2, y_1', y_2', z_1', \bar{y}_1', \bar{y}_2', \bar{z}_1', \lambda, \mu) \\ &\quad = F(x, y_1, y_2, z_1, \bar{z}_2, \bar{y}_1', \bar{y}_2', \bar{z}_1', \lambda, \mu) \\ &\quad\quad - F(x, y_1, y_2, z_1, z_2, y_1', y_2', z_1', \lambda, \mu) \\ &\quad\quad - \sum_{i=1}^{2} (\bar{y}_i' - y_i') F_{y_i'}(x, y_1, y_2, z_1, z_2, y_1', y_2', z_1', \lambda, \mu) \\ &\quad\quad - (\bar{z}_1' - z_1') F_{z_1'}(x, y_1, y_2, z_1, z_2, y_1', y_2', z_1', \lambda, \mu) \end{aligned}$$

which yields for our case

$$E(z_2, \bar{z}_2, y_2', \bar{y}_2') = \bar{z}_2(\sqrt{\bar{y}_2'} - \sqrt{y_2'}) + \frac{z_2}{2\sqrt{y_2'}} (y_2' - \bar{y}_2').$$

We note right away that

$$E(z_2, \bar{z}_2, y_2', \bar{y}_2') \equiv 0 \qquad \text{for all} \quad \bar{y}_2' = y_2' \tag{13}$$

and that E has a saddle point at $\bar{y}_2' = y_2'$, $\bar{z}_2' = z_2'$, because

$$\begin{aligned} &E_{\bar{z}_2} = \sqrt{\bar{y}_2'} - \sqrt{y_2'}, \qquad E_{\bar{y}_2'} = \frac{\bar{z}_2}{2\sqrt{\bar{y}_2'}} - \frac{z_2}{2\sqrt{y_2'}} \\ &E_{\bar{z}_2\bar{z}_2} = 0, \qquad E_{\bar{z}_2\bar{y}_2'} = \frac{1}{2\sqrt{\bar{y}_2'}}. \end{aligned}$$

(See Fig. 1 which was produced by N. J. Rose on a MacIntosh with a laser printer, using the MacFunction program.)

Hence, the Weierstrass condition cannot possibly be satisfied and we need not bother with the construnction of a Mayer field for that purpose.

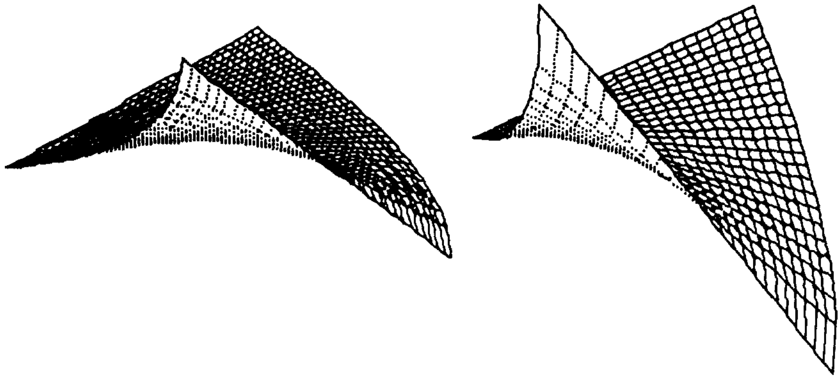

FIG. 1. The excess function at $y_2' = 4$, $z_2 = 4$, viewed from two different vantage points.

Equation (13) strongly suggests that the integral is invariant under variations of the extremal that leave y_2 alone. To confirm this, using (13), we would have to construct a Mayer field to validate the transformation of the total variation. We will take a more direct approach in the next section by constructing a *linear* variation of the extremal without changing y_2 and demonstrating the invariance of the integral under such variations. It will develop that these variations are not extremals which will lead, in turn, to the result that the integral does not even attain a weak extremum.

4. Linear Variations and the Impossibility of a Weak Extremum

Let C_0 denote an extremal that satisfies given boundary conditions and let C_ε denote a linear variation of C_0 which leaves y_2 unchanged. That there are such linear variations may be seen as follows:

Let η denote a function with a continuous derivative with $\eta(0) = \eta(a) = 0$ and let

$$\bar{y}_1(x, \varepsilon) = y_1(x) + \varepsilon\eta(x),$$

where ε denotes a parameter. Clearly,

$$\bar{y}_1(0, \varepsilon) = y_1(0), \qquad \bar{y}_1(a, \varepsilon) = y_1(a).$$

Since $\bar{y}_1$ has to satisfy (4), we have

$$\bar{y}_1' = y_1' + \varepsilon\eta' = \bar{z}_2 \cos y_2$$

and hence,

$$\bar{z}_2 = \frac{y_1'}{\cos y_2} + \varepsilon \frac{\eta'}{\cos y_2} = z_2 + \varepsilon \frac{\eta'}{\cos y_2}.$$

Then,

$$\bar{z}_1' = \bar{z}_2 \sin y_2 = z'_1 + \varepsilon\eta' \tan y_2.$$

We have to impose an additional condition on η to ensure that $\bar{z}_1(0, \varepsilon) = z_1(0)$, $\bar{z}_1(a, \varepsilon) = z_1(a)$. This is possible as the following argument reveals: With

$$\bar{z}_1(x) = z_1(x) + \varepsilon \int_0^x \eta'(t) \tan y_2(t)\, dt$$

we have to have

$$\int_0^a \eta' \tan y_2\, dx = 0. \tag{14}$$

Since

$$\int_0^a \eta' \tan y_2\, dx = \eta(x) \tan y_2(x) \big|_0^a - \int_0^a \frac{\eta}{\cos^2 y_2}\, y_2'\, dx$$

we have to impose the additional condition

$$\int_0^a \frac{\eta}{\cos^2 y_2}\, y_2'\, dx = 0.$$

From (8), $y_2'(x) > 0$ and this condition is easily satisfied. (Let, for example, $\eta(x) = x(x - \xi)(x - a)$. Then

$$H(\xi) = \int_0^a \eta \frac{y_2'}{\cos y_2}\, dx$$

is a continuous function with $H(0) < 0$, $H(a) > 0$ and hence, $H(\xi_0) = 0$ for some $\xi_0 \in (0, a)$.)

We have finally arrived at the linear variation C_ε of C_0

$$\begin{aligned} \bar{y}_1 &= y_1 + \varepsilon\eta \\ \bar{y}_2 &= y_2 \\ \bar{z}_1 &= z_1 + \varepsilon \int_0^x \eta' \tan y_2\, dt \\ \bar{z}_2 &= z_2 + \varepsilon \frac{\eta'}{\cos y_2}, \end{aligned} \tag{15}$$

where

$$\eta(0)=\eta(a)=0 \qquad \text{and} \qquad \int_0^a \eta \frac{y_2'}{\cos^2 y_2}\,dx = -\int_0^a \eta' \tan y_2\,dx = 0. \tag{16}$$

Because of (8) and (16)

$$\begin{aligned}\frac{d}{d\varepsilon} I_{C_\varepsilon} &= \frac{d}{d\varepsilon}\int_0^a \bar{z}_2 \sqrt{y_2'}\,dx = \frac{d}{d\varepsilon}\int_0^a \left(z_2+\varepsilon\frac{\eta'}{\cos y_2}\right)\sqrt{y_2'}\,dx \\ &= \int_0^a \frac{\eta'}{\cos y_2}(\lambda\cos y_2+\mu\sin y_2)\,dx \\ &= \lambda\eta\,|_0^a+\mu\int_0^a \eta'\tan y_2\,dx = 0\end{aligned}$$

and hence

$$I_{C_\varepsilon} = I_{C_0} \tag{17}$$

for all C_ε in (15).

Equation (16) is also responsible for the fact that C_ε, for $\varepsilon \neq 0$, is not an extremal because $\bar{z}_2$, as defined in (15), does not satisfy the Euler–Lagrange equation (7) with λ, μ constant, and with y_2 satisfying (8), as we will demonstrate in what is to follow: Since z_2 does satisfy (7) and since (7) is linear in z_2, we only need to show that

$$\frac{\eta'}{\cos y_2}$$

when substituted for z_2 does not satisfy (7). If it were to satisfy (7), then by (11),

$$\frac{\eta'}{\cos y_2} = \beta\sqrt{1+[(\lambda^2+\mu^2)(x+\alpha)]^2}$$

and by (10)

$$\eta' = \beta(\lambda - \mu(\lambda^2+\mu^2)(x+\alpha))$$

and hence

$$\eta = \beta\left(\lambda x - \mu(\lambda^2+\mu^2)\frac{(x+\alpha)^2}{2}\right)+\gamma \qquad \text{for arbitrary } \gamma.$$

In order to satisfy (16), we have to have $\eta \equiv 0$ which takes us back to C_0.

If I_{C_ε} were a weak minimum, then C_ε would, by necessity, be an extremal. It is not. Hence, one can find in any strong neighborhood of C_ε a C_ε^* such that $I_{C_\varepsilon^*} < I_{C_\varepsilon}$. Since there is a C_ε in any strong neighborhood of C_0, it follows that C_0 *does not yield a weak minimum.* An analogous argument reveals that it does not yield a weak maximum either.

5. Concluding Remarks

There are, of cource, quasi-regular Lagrange problems where the integral attains a strong minimum. Take, for example, the problem of finding the minimum of the integral

$$\int_0^1 (y_1'^2 + y_3^2)\, dx$$

with the boundary conditions $y_1(0) = y_2(0) = 0$, $y_1(1) = y_2(1) = 1$ and the constraining equation $y_2' - y_3 = 0$. Then, $y_1 = y_2 = x$, $y_3 = 1$, $\lambda = 2$ yields the strong minimum 2.

To return to Irrgang's problem with the integrand $\sqrt{\tau}$: In the original formulation, there are 14 unknown functions, namely the components of the position-, tangent-, normal-, and binormal-vector, and the curvature and torsion. Since the derivatives of the latter two are missing, the problem is singular. Were one to extend the concept of quasi-regularity in Radon's spirit to problems of this type, Irrgang's problem would still not be quasi-regular. Weierstrass' necessary condition [6] is, of course, applicable but inconclusive. Due to the nature of the constraints and because κ' and τ' are missing, the excess function is identically zero for all lineal elements of the extremal and all admissible comparison values for the derivatives. (In fact, the only admissible comparison values are the values of the derivatives along the extremals themselves.) While this does not preclude the possibility of a strong extremum, it does not permit the drawing of any conclusions. If one goes back to the geometric definition of torsion as a measure of the rapidity with which a curve twists in space, rather than rely on one of the coordinate-dependent representations with limited validity, one can, in fact, find an absolute minimum. (See (7).)

References

1. J. Radon, Ueber das Minimums des Integrals $\int F(x, y, \Theta, \kappa)\, ds$, *S.-B. Kaiserl. Akad. Wiss. Wien, Abt. 2a (Math)* **119** (1990), 1257–1326.
2. W. Blaschke, "Vorlesungen ueber Differentialgeometrie I." pp. 51–55, Chelsea, New York, 1967.

3. R. IRRGANG, Ein singulaeres bewegungsinvariantes Variationsproblem, *Math. Z.* **37** (1933), 381–401.
4. J. RADON, Singulaere Variationsprobleme, *J. Ber. Deutsch. Math. Verein.* **47** (1937), 220–232.
5. H. SAGAN, "Introduction to the Calculus of Variations," McGraw–Hill, New York, 1969.
6. E. J. MCSHANE, On multipliers for lagrange problems, *Amer. J. Math.* **61** (1939), 809–819.
7. H. SAGAN, A singular solution to Irrgang's Problem, Proc. X COBEM 1989, to appear.